

s -points in 3d acoustical scattering

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Abstract

The notion of s -points has been introduced by the authors (SIAM JMA, 39 (2008), 1821–1850) in connection with the control problem for the dynamical system governed by the 3d acoustical equation $u_{tt} - \Delta u + qu = 0$ with a real potential $q \in C_0^\infty(\mathbb{R}^3)$ and controlled by incoming spherical waves. In the generic case, this system is controllable in the relevant sense, whereas $a \in \mathbb{R}^3$ is called a s -point (we write $a \in \Upsilon_q$) if the system with the shifted potential $q_a = q(\cdot - a)$ is *not controllable*. Such a lack of controllability is related to the subtle physical effect: in the system with the potential q_a there exist the finite energy waves vanishing in the past and future cones simultaneously. The subject of the paper is the set Υ_q : we reveal its relation to the factorization of the S -matrix, connections with the discrete spectrum of the Schrödinger operator $-\Delta + q$ and the jet degeneration of the polynomially growing solutions to the equation $(-\Delta + q)p = 0$.

0 Introduction

0.1 Dynamical system

An acoustical scattering problem is the system of the form

$$u_{tt} - \Delta u + qu = 0, \quad (x, t) \in \mathbb{R}^3 \times (-\infty, \infty) \quad (0.1)$$

$$u|_{|x| < -t} = 0, \quad t < 0 \quad (0.2)$$

$$\lim_{s \rightarrow \infty} su((s + \tau)\omega, -s) = f(\tau, \omega), \quad (\tau, \omega) \in [0, \infty) \times S^2, \quad (0.3)$$

where $u = u^f(x, t)$ is a solution (*wave*), $q = q(x)$ is a real valued smooth¹ compactly supported function (*potential*), and $f \in \mathcal{F} := L_2([0, \infty); L_2(S^2))$

¹everywhere in the paper "smooth" means C^∞ -smooth.

is a *control*. With the system one associates the *control operator* $W : \mathcal{F} \rightarrow \mathcal{H} := L_2(\mathbb{R}^3)$, $Wf := u^f(\cdot, 0)$ and the subspaces: $\mathcal{U} := \text{Ran } W$ (*reachable set*), $\mathcal{D} := \mathcal{H} \ominus \mathcal{U}$ (*defect subspace*), $\mathcal{N} := \text{Ker } W \subset \mathcal{F}$ (*null control subspace*). The relations $0 \leq \dim \mathcal{D} = \dim \mathcal{N} < \infty$ hold [2]. If $\mathcal{D} = \{0\}$, the system (0.1)–(0.3) is said to be controllable²; the case of $\mathcal{D} \neq \{0\}$ is referred to as a lack of controllability.

0.2 s -points

As is shown in [2], the case $\dim \mathcal{D} \neq 0$ is realizable and corresponds to the curious effect: for $f \in \mathcal{N}$, the function $w^f(x, t) := \int_0^t u^f(x, s) ds$ is a finite energy solution of (0.1), which satisfies $w^f(\cdot, -t) = w^f(\cdot, t)$ and

$$w^f|_{|x| < |t|} = 0, \quad (0.4)$$

i.e., describes the wave vanishing in the past and future cones simultaneously. This wave comes from infinity, stops at the moment $t = 0$ ³, and then return back to infinity along the same trajectory. Such a behavior motivates to call it a *reversing wave* and regard the coordinate system origin $x = 0$ as a point, which is able to stop incoming waves ('stop point'). Looking for such points in the space, we change the coordinates $x \mapsto x - a$ and deal with the system (0.1)–(0.3) with the shifted potential $q_a := q(x - a)$, the objects corresponding to q_a being labelled with the subscript a . An $a \in \mathbb{R}^3$ is said to be a *s -point of the potential q* if $\mathcal{D}_a \neq \{0\}$ holds. By Υ_q we denote the set of such points.

To the best of our knowledge, s -points is something new in the acoustical scattering and the set Υ_q is worth studying for its own sake. The goal of our paper is to originate such a study: we reveal certain relations between Υ_q and the known objects of the scattering theory.

0.3 Results

Evolution of the system (0.1)–(0.3) is governed by the Schrödinger operator $H := -\Delta + q$ in \mathcal{H} , $\text{Dom } H = H^2(\mathbb{R}^3)$. This operator may have a finite discrete spectrum $\sigma_{\text{disc}}(H) \subset (-\infty, 0]$ (see, e.g., [8]) and in the paper, for the sake of simplicity, we accept

²Note that the unperturbed system (with $q = 0$) is controllable.

³We mean $w_t^f(\cdot, 0) = 0$.

Convention 1 *The potential q is such that the equation*

$$(-\Delta + q)\varphi = 0 \quad \text{in } \mathbb{R}^3 \quad (0.5)$$

has not nonzero solutions, which satisfy $\varphi(x) \rightarrow 0$ as $|x| \rightarrow 0$.

In physical terms, this means that the hamiltonian H has neither bound nor semi-bound state at the zero energy level (see, e.g., [6]). If exists, such a level can be removed by arbitrarily small perturbation of the potential q , so that the convention is not restrictive and we deal with a generic case.

Our results are the following.

- Let $S_a(k)$, $k \in \mathbb{R}$ be the Schrödinger S -matrix of the potential q_a . In the series of papers cited in his book [6], R.Newton introduced the Riemann–Hilbert factorization of the form

$$S_a(k) = \Pi_a^-(k) [S_a^-(k)S_a^+(k)] \Pi_a^+(k), \quad k \in \mathbb{R}, \quad (0.6)$$

where Π_a^\pm are the Blaschke-type rational operator-valued functions, S_a^\pm are the operator-valued functions holomorphic and boundedly invertible in the half-planes $\{k \in \mathbb{C} \mid \pm \Im k > 0\}$ respectively. This representation is used for solving the inverse problem that is to determine the potential q from the scattering data: it is shown that the knowledge of the family $\{S_a^\pm\}_{a \in \mathbb{R}^3}$ enables one to recover q . In the mean time, in the mentioned papers it is assumed that such a factorization is realizable for any $a \in \mathbb{R}^3$. *We show that for $a \in \Upsilon_q$ the representation (0.6) fails, i.e., the factorization is impossible*⁴.

- In [2] we suggested that the presence of s -points is connected with the discrete spectrum of the operator H : the conjecture was that $\Upsilon_q \neq \emptyset$ is equivalent to $\sigma_{\text{disc}}(H) \neq \emptyset$. Here this conjecture is partially justified as follows.

Let a point $a \in \mathbb{R}^3$, an integer $m \geq 1$, and a function $h \in \mathcal{H}$ be such that

$$(-\Delta + q)^m h = 0 \quad \text{in } \mathbb{R}^3 \setminus \{a\} \quad (0.7)$$

holds, whereas $(-\Delta + q)^{m-1}h$ does not vanish identically. As is shown in [2], such an a is necessarily an s -point; by $\Upsilon_q^m \subset \Upsilon_q$ we denote the set of these points and specify m as the order of a . *We prove that $\Upsilon_q^1 \neq \emptyset$ is equivalent to $\sigma_{\text{disc}}(H) \neq \emptyset$.*

⁴However, we do not claim that this fact cancels the determination of the potential by R.Newton's procedure.

- We show that the points $a \in \Upsilon_q^{\text{fin}} := \cup_{m \geq 1} \Upsilon_q^m$ ⁵ can be specified as the *jet degeneration points of polynomially growing solutions* to the equations $(-\Delta + q)p = 0$ in \mathbb{R}^3 . By this, typically the set Υ_q^{fin} consists of s -surfaces. If the potential is radially symmetric, i.e., $q = q(|x|)$, the s -surfaces are spheres and their position in the space can be studied in more detail.

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1 s -points and R.Newton's factorization

1.1 Scattering matrix

Recall the basic definitions and facts (see, e.g., [6]).

The Schrödinger *scattering operator* of the pair $H_0 = -\Delta$, $H = -\Delta + q$ is $S : \mathcal{H} \rightarrow \mathcal{H}$, $S := W_+^* W_-$, where $W_{\pm} := s - \lim_{t \rightarrow \pm\infty} e^{-itH} e^{itH_0}$ are the *wave operators*.

Let $(Fy)(p) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{ipx} y(x) dx$ be the Fourier transform. The operator $\tilde{S} := F S F^{-1}$ is of the form

$$\left(\tilde{S}v \right) (p) = (S(k) [v(k, \cdot)]) (\theta),$$

where $v(k, \omega) := v(k\omega)$, $k := |p|$, $\theta := \frac{p}{|p|}$, $\omega, \theta \in S^2 := \{p \in \mathbb{R}^3 \mid |p| = 1\}$, whereas $S(k)$ is an operator acting in $L_2(S^2)$ on the angle variable by the rule

$$(S(k)g)(\theta) := \int_{S^2} s(\theta, \omega; k) g(\omega) d\omega,$$

depending on k as a parameter and being called the *S-matrix* of the potential q . So, we regard the S-matrix as an operator-valued function of $k > 0$, the values being taken in the bounded operator algebra $\mathcal{B}(L_2(S^2))$. The kernel $s(\theta, \omega; k)$ is a distribution of the form $\delta(\theta - \omega) + \tilde{s}(\theta, \omega; k)$ with a smooth \tilde{s} , so that each $S(k)$ is of the form "identity + compact operator". Moreover, each $S(k)$ is a unitary operator. Also, the well-known high energy asymptotic $S(k) \rightarrow \mathbb{I}$ as $k \rightarrow \infty$ holds.

⁵Presumably, Υ_q^{fin} exhausts Υ_q but it is still a conjecture.

In what follows, we assume $S(k)$ extended to $k < 0$ by $S(k) := S^*(-k)$. The index of the S-matrix is defined by

$$\text{ind} S := \frac{1}{4\pi i} [\ln \det S(k)] \Big|_{k=-\infty}^{k=+\infty};$$

by the Levinson theorem it is equal to the total multiplicity of the discrete spectrum $\sigma_{\text{disc}}(H)$. The representation

$$S(k) = \Pi^-(k) S^{\text{red}}(k) \Pi^+(k), \quad k \in \mathbb{R} \quad (1.1)$$

is valid, where S^{red} is the so-called *reduced S-matrix*, which satisfies $\text{ind} S^{\text{red}} = 0$, and Π is a $\mathcal{B}(L_2(S^2))$ -valued function of the form

$$\Pi^\pm(k) = \prod_{n=1}^l \left(\mathbb{I} + \frac{2ik}{k - ik_n} B_n^\pm \right), \quad k \in \mathbb{R}$$

with the certain projections $B_n^\pm = (\cdot, \psi_n^\pm)_{L_2(S^2)} \psi_n^\pm$, where $\psi^+(\omega) = \psi^-(-\omega)$, and $k_n > 0$ such that $-k_n^2 \in \sigma_{\text{disc}}(H)$ (see [6]).

1.2 Factorization by R.Newton

In solving the scattering inverse problem by R.Newton, the following Riemann-Hilbert type representation of the reduced S-matrix plays the key role:

$$S^{\text{red}}(k) = S^-(k) S^+(k), \quad k \in \mathbb{R}, \quad (1.2)$$

where S^\pm are the $\mathcal{B}(L_2(S^2))$ -valued operator-functions, which are holomorphic, bounded, and boundedly invertible in the complex half-planes $\mathbb{C}^\pm = \{k \in \mathbb{C} \mid \pm \Im k > 0\}$ respectively. In more detail, to recover the potential q via its S-matrix, one needs

- for a fixed $a \in \mathbb{R}^3$, to find the S-matrix S_a of the shifted potential q_a that can be done in a simple and explicit way
- to determine S_a^{red} from (1.1) and represent $S_a^{\text{red}} = S_a^- S_a^+$ by (1.2)
- by varying a , to collect the families $\{S_a^\pm\}_{a \in \mathbb{R}^3}$; each of them determines the potential q .

However, in [6] the author assumes that the second step can be fulfilled for all $a \in \mathbb{R}^3$. The following result shows that such an assumption may be invalid.

Theorem 1 *If $a \in \Upsilon_q$ then the representation (1.2) for S^{red}_a does not hold, i.e., the factorization $S^{\text{red}}_a = S^-_a S^+_a$ is impossible.*

We postpone the proof till section 1.4 and begin with preliminaries concerning to the well-known facts of the Lax-Phillips theory.

1.3 On Lax-Phillips scheme

The Cauchy problem for the acoustical equation is the system

$$v_{tt} - \Delta v + qv = 0, \quad (x, t) \in \mathbb{R}^3 \times (-\infty, \infty) \quad (1.3)$$

$$v|_{t=0} = \varphi, \quad v_t|_{t=0} = \psi \quad \text{in } \mathbb{R}^3 \quad (1.4)$$

with the finite energy data $d := \{\varphi, \psi\} \in D := H^1(\mathbb{R}^3) \times L_2(\mathbb{R}^3)$; the solution is denoted by $v = v^d(x, t)$. A peculiarity of the acoustical scattering is that the *energy form*

$$E[d, d'] := \int_{\mathbb{R}^3} \psi \psi' + \nabla \varphi \cdot \nabla \varphi' + q \varphi \varphi',$$

which the set of data is equipped with, is indefinite. However, the form turns out to be positive definite on the *absolutely continuous subspace* $D_{\text{ac}} := D \ominus_E D_{\text{disc}}$, where D_{disc} is the finite dimensional subspace spanned on the data $\{\varphi_k, \pm k \varphi_k\}$ such that $-k^2 \in \sigma_{\text{disc}}(H)$ and $H \varphi_k = -k^2 \varphi_k$. By P_{ac} we denote the projection on the first summand of the decomposition $D = D_{\text{ac}} \oplus_E D_{\text{disc}}$.

In the framework of the Lax-Phillips theory [5], there is a certain freedom in the choice of the pair of the *incoming* and *outgoing subspaces* D^\pm . Once such a choice made, for each $d \in D$ the corresponding *incoming* and *outgoing spectral representatives* \tilde{d}_\mp , which are the S^2 -valued functions of $k \in \mathbb{R}$ of the class $L_2(\mathbb{R}; L_2(S^2))$, do appear. In the spectral representation, the scattering process is described by the *Lax-Phillips scattering operator* $\tilde{S} : \tilde{d}_- \mapsto \tilde{d}_+$, which acts in $L_2(\mathbb{R}; L_2(S^2))$ by

$$(\tilde{S}g)(k) = \tilde{S}(k)g(k), \quad k \in \mathbb{R},$$

where $\tilde{S}(\cdot)$ is a $\mathcal{B}(L_2(S^2))$ -valued operator-function (*the Lax-Phillips S -matrix*). Two possible variants of the choice are the following.

1. Assign a $d \in D$ to the subspace $D_0^- \subset D$ if $v^d|_{|x|<-t} = 0$ for $t < 0$ and the subspace $D_0^+ \subset D$ if $v^d|_{|x|<t} = 0$ for $t > 0$. The subspaces

$$D^\mp := P_{ac} D_0^\mp$$

constitute an incoming/outgoing pair [5]. By \tilde{S} and $\tilde{S}(\cdot)$ we denote the corresponding scattering operator and its S -matrix.

2. One can reduce the incoming/outgoing subspaces to the smaller ones

$$D_{\text{red}}^\mp := D_0^\mp \cap D_{ac} \subset D^\mp.$$

By \tilde{S}^{red} and $\tilde{S}^{\text{red}}(\cdot)$ we denote the corresponding operator and S -matrix.

The important fact, which relates the quantum scattering and acoustical scattering objects and follows from the above accepted definitions, is that the equalities

$$\tilde{S}(k) = S(k), \quad \tilde{S}^{\text{red}}(k) = S^{\text{red}}(k), \quad k \in \mathbb{R} \quad (1.5)$$

hold (see [5], chapter VI, part 2, the relation (3.4)). Note in addition that (1.5) and (1.1) imply

$$\tilde{S}(k) = \Pi^-(k) \tilde{S}^{\text{red}}(k) \Pi^+(k), \quad k \in \mathbb{R},$$

whereas the factors $\Pi^\mp(\cdot)$ can be also interpreted in terms of the Lax-Phillips theory as S -matrices corresponding to the certain choice of the incoming/outgoing subspaces.

Let

$$L_2(\mathbb{R}; L_2(S^2)) = \mathcal{H}^- \oplus \mathcal{H}^+ \quad (1.6)$$

be the decomposition on the Hardy subspaces \mathcal{H}^\mp which consist of functions holomorphic in $\{k \in \mathbb{C} \mid \mp \Im k > 0\}$ respectively; by P^\mp we denote the projections on \mathcal{H}^\mp . As is known, if d belongs to the incoming (outgoing) subspace then its spectral representative lies in \mathcal{H}^- (\mathcal{H}^+). Representing the scattering operator in the matrix form in accordance with (1.6), one has

$$\tilde{S}^{\text{red}} = \begin{pmatrix} P^- \tilde{S}^{\text{red}} P^- & P^- \tilde{S}^{\text{red}} P^+ \\ P^+ \tilde{S}^{\text{red}} P^- & P^+ \tilde{S}^{\text{red}} P^+ \end{pmatrix}. \quad (1.7)$$

1.4 Proof of Theorem 1

Assume that $a \in \mathbb{R}^3$ is a s -point of the potential q . As is evident, by shifting the coordinate system, one can provide $a = 0$. So, let $0 \in \Upsilon_q$.

Let w^f be a reversing wave (see section 0.2), $d_0 := \{w^f(\cdot, 0), 0\}$ its Cauchy data. By (0.4), one has $d_0 \in D^- \cap D^+$; moreover, the results [2] on the stability of trajectories easily imply $d_0 \in D_{\text{red}}^- \cap D_{\text{red}}^+$. The latter leads to $\tilde{d}_0^- \in \mathcal{H}^-$ and $\tilde{d}_0^+ \in \mathcal{H}^+$. Therefore,

$$\tilde{S}^{\text{red}} \tilde{d}_0^- = \tilde{d}_0^+ \in \mathcal{H}^+$$

and, applying the projection P^- , one gets $P^- \tilde{S}^{\text{red}} \tilde{d}_0^- = P^- \tilde{S}^{\text{red}} P^- \tilde{d}_0^- = 0$. Thus, we see that $\text{Ker } P^- \tilde{S}^{\text{red}} P^- \neq \{0\}$, i.e., the block $P^- \tilde{S}^{\text{red}} P^-$ in (1.7) is not invertible.

By (1.5), the block $P^- S^{\text{red}} P^-$ is also not invertible. However, as is well known, such an invertibility is the necessary condition that provides the Riemann-Hilbert factorization (1.2) (see, e.g., [6]). Hence, in the case of $a \in \Upsilon_q$, to represent S^{red}_a in the form (1.2) is impossible. \square

In addition, note the following. Although the representation (1.2) fails for $a \in \Upsilon_q$, by very general results of the Riemann-Hilbert theory one can factorize the reduced S-matrix as

$$S^{\text{red}}_a(k) = \check{\Pi}_a^-(k) [\check{S}_a^-(k) \check{S}_a^+(k)] \check{\Pi}_a^+(k), \quad k \in \mathbb{R} \quad (1.8)$$

with the Blaschke-type operator functions $\check{\Pi}_a, \check{\Pi}_a^*$, which have the certain complex poles, and \check{S}_a^\pm holomorphic and boundedly invertible in $\{k \in \mathbb{C} \mid \pm \Im k > 0\}$ [3]. Moreover, it is the presence of the Blaschke factors, which renders the representation (1.2) impossible. In this connection, the intriguing questions arises: What is the physical meaning of the factorization (1.8) and the corresponding Blaschke poles? Can one characterize them in terms of the Lax-Phillips theory? The questions are open.

2 s -points and discrete spectrum

2.1 Theorem 2

Assume that for a point $a \in \mathbb{R}^3$ there exist a function $h \in \mathcal{H}$ and an integer $k \geq 1$ such that

$$(-\Delta + q)^k h = 0 \quad \text{in } \mathbb{R}^3 \setminus \{a\}$$

and let m be the minimal of such k 's. As is shown in [2], this a is necessarily an s -point, whereas m is specified as the order of a . By $\Upsilon_q^m \subset \Upsilon_q$ we denote the set of s -points of the order m .

Remark In the light of these definitions, Convention 1 is motivated as follows. If $0 \in \sigma_{\text{disc}}(H)$ and $\varphi_0 \in \mathcal{H}$ is a zero energy level eigenfunction, then $(-\Delta + q)\varphi_0 = 0$ *everywhere* in \mathbb{R}^3 and we are forced to accept $\Upsilon_q^1 = \Upsilon_q = \mathbb{R}^3$. The convention excludes such a degeneration.

The following result partially explains why s -points do appear: one of the reasons is the presence of the discrete spectrum of the operator H .

Theorem 2 $\Upsilon_q^1 \neq \emptyset$ holds if and only if $\sigma_{\text{disc}}(H) \neq \emptyset$.

The rest of section 2 is the proof of the theorem.

2.2 Green function

We define the *Green function* of the operator H as a $L_2^{\text{loc}}(\mathbb{R}^3)$ -solution of the integral equation

$$G(x, y) = \frac{1}{4\pi|x-y|} - \int_{\mathbb{R}^3} \frac{q(s)}{4\pi|x-s|} G(s, y) ds, \quad x \in \mathbb{R}^3, \quad (2.1)$$

where $y \in \mathbb{R}^3$ is a parameter. This definition is correct since this equation is uniquely solvable. Indeed, otherwise there is a nonzero solution φ of the homogeneous equation

$$\varphi(x) = - \int_{\mathbb{R}^3} \frac{\varphi(s)}{4\pi|x-s|} q(s) ds, \quad x \in \mathbb{R}^3.$$

As is easy to see, such a solution satisfies (0.5) and vanishes at infinity that is forbidden by Convention 1. To establish the solvability, one can reduce the equation to a large enough ball $B \supset \text{supp } q$, prove the solvability there, and then extend the solution to $\mathbb{R}^3 \setminus B$ as a r.h.s. of (2.1).

Also, the integral equation implies the symmetry property $G(x, y) = G(y, x)$, $x \neq y$. From (2.1) one derives that in the sense of distributions on $C_0^\infty(\mathbb{R}^3)$ the function G satisfies

$$(-\Delta + q)G(\cdot, y) = \delta_y, \quad (2.2)$$

where δ_y is the Dirac measure supported in y .

Lemma 1 *The asymptotic*

$$G(x, y) \underset{|x| \rightarrow \infty}{=} \frac{\Phi(y)}{4\pi|x|} + O\left(\frac{1}{|x|^2}\right), \quad y \in \mathbb{R}^3 \quad (2.3)$$

holds, where Φ is a smooth function obeying

$$(-\Delta + q)\Phi = 0 \quad \text{in } \mathbb{R}^3 \quad (2.4)$$

$$\Phi(x) \rightarrow 1 \quad \text{as } |x| \rightarrow \infty, \quad (2.5)$$

the asymptotics (2.3) and (2.5) being uniform w.r.t. $\frac{x}{|x|} \in S^2$.

Proof Fix x in the equation (2.1) and tend $|y| \rightarrow \infty$. Using the symmetry property and looking for the solution in the form (2.3), we have

$$\frac{\Phi(x)}{4\pi|y|} + O\left(\frac{1}{|y|^2}\right) = \frac{1}{4\pi|y|} + O\left(\frac{1}{|y|^2}\right) - \int_{\mathbb{R}^3} \frac{q(s)}{4\pi|x-s|} \left[\frac{\Phi(s)}{4\pi|y|} + O\left(\frac{1}{|y|^2}\right) \right] ds$$

that easily leads to the integral equation

$$\Phi(x) = 1 - \int_{\mathbb{R}^3} \frac{q(s)}{4\pi|x-s|} \Phi(s) ds, \quad x \in \mathbb{R}^3. \quad (2.6)$$

The latter equation implies (2.4), (2.5), and justifies (2.3). \square

Now we are ready for proving Theorem 2.

2.3 Necessity

Let $\sigma_{\text{disc}}(H) \neq \emptyset$ and $H\chi = -k_0^2\chi$ for $-k_0^2 = \inf \sigma_{\text{disc}}(H)$, so that χ is a ground state of the operator H . As is well known, $-k_0^2$ is an ordinary eigenvalue of H , whereas the eigenfunction χ behaves as $\chi(x) \underset{|x| \rightarrow \infty}{\sim} e^{-k_0|x|}$

and can be chosen positive: $\chi > 0$ *everywhere* in \mathbb{R}^3 . Integrating by parts, we have

$$0 = \langle \text{see (2.4)} \rangle = \int_{\mathbb{R}^3} \chi(x) [(-\Delta + q)\Phi](x) dx = -k_0^2 \int_{\mathbb{R}^3} \chi(x)\Phi(x) dx.$$

Hence, Φ has to change sign and there is an $a \in \mathbb{R}^3$ such that $\Phi(a) = 0$. As result, we conclude that $G(\cdot, a) \in \mathcal{H}$ (see (2.3)), whereas $(-\Delta + q)G(\cdot, a) = 0$ in $\mathbb{R}^3 \setminus \{a\}$ (see (2.2)). Thus, $a \in \Upsilon_q^{-1}$ and, hence, $\Upsilon_q^{-1} \neq \emptyset$.

2.4 Sufficiency

The sufficiency will be proved by contradiction. Assume that $\sigma_{\text{disc}}(H) = \emptyset$ but $\Upsilon_q^{-1} \neq \emptyset$ and, hence, there is a point $a \in \mathbb{R}^3$ and a function $h \in \mathcal{H}$ such that $(-\Delta + q)h = 0$ holds in $\mathbb{R}^3 \setminus \{a\}$.

Show that $\Phi(a) = 0$. Indeed, considering $(-\Delta + q)h$ as a distribution on $C_0^\infty(\mathbb{R}^3)$, we see that it is supported at $x = a$. Such a distribution has to be a linear combination of the Dirac measure derivatives:

$$(-\Delta + q)h = \sum_{|j|=0}^N \alpha_j D_x^j \delta_a = \sum_{|j|=0}^N \alpha_j (-1)^{|j|} D_a^j \delta_a,$$

where $j = \{j_1, j_2, j_3\}$ is a multi-index, $|j| = j_1 + j_2 + j_3$ and $D_x^j = \partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \partial_{x_3}^{j_3}$ is a differentiation, α_j are constants (see, e.g. [9]). Therefore, by (2.2), the function h has to be of the form

$$h(x) = \sum_{|j|=0}^N \alpha_j (-1)^{|j|} D_a^j G(x, a) + \tilde{h}(x), \quad (2.7)$$

with a \tilde{h} satisfying $-(\Delta + q)\tilde{h} = 0$ as a distribution. Hence, \tilde{h} is a smooth function and

$$-(\Delta + q)\tilde{h} = 0 \quad \text{in } \mathbb{R}^3 \quad (2.8)$$

holds in the classical sense. As can be shown from (2.1), for $|j| \geq 1$ the derivatives $D_a^j G(\cdot, a)$ are not square-summable near $x = a$. Therefore, the only option for the sum in (2.7) (and, hence, for h) to belong to \mathcal{H} (to be square-summable near $x = a$) is $\alpha_j = 0$ for $|j| \geq 1$ and we get

$$h(x) = \alpha_0 G(x, a) + \tilde{h}(x) \quad \text{in } \mathbb{R}^3 \setminus \{0\}.$$

For large enough $|x| > \text{diam supp } q$, we have $\Delta h = 0$ and know that $h \in \mathcal{H}$. Such a function vanishes as $|x| \rightarrow \infty$. Hence, by (2.3), the function \tilde{h} also tends to zero as $|x| \rightarrow \infty$ and, in the same time, satisfies (2.8). By Convention 1, one has $\tilde{h} = 0$. Thus, h is proportional to the Green function $G(\cdot, a)$, whereas $G(\cdot, a) \in \mathcal{H}$ implies $\Phi(a) = 0$ by (2.3).

Now we apply the perturbation theory arguments. Consider an operator family $\{-\Delta + \varepsilon q\}_{\varepsilon \in [0,1]}$; let Φ^ε be the corresponding analog of the function $\Phi \equiv \Phi^1$ that appears in the asymptotic (2.3). Note that for $\varepsilon = 0$ one has

$\Phi^0 = 1$. As is easy to see from the integral equation (2.1), for small ε the inequality $\Phi^\varepsilon(\cdot) > 0$ holds everywhere in \mathbb{R}^3 , whereas

$$\Phi^\varepsilon(x) \rightarrow 1 \quad \text{as } |x| \rightarrow \infty \quad (2.9)$$

uniformly w.r.t. ε .

Since Φ^ε depends on ε continuously ⁶, there is the minimal $\varepsilon_0 \in (0, 1]$ such that the function Φ^{ε_0} satisfies $\Phi^{\varepsilon_0} \geq 0$ and, in the mean time, has zeros. Shifting (if necessary) the origin of coordinates, assume that $\Phi^{\varepsilon_0}(0) = 0$ and recall that Φ^{ε_0} satisfies $(-\Delta + \varepsilon_0 q) \Phi^{\varepsilon_0} = 0$ in \mathbb{R}^3 .

Fix a positive r ; let $B_r := \{x \in \mathbb{R}^3 \mid |x| < r\}$ and $S_r := \partial B_r$. Integration by parts implies

$$\begin{aligned} \varepsilon_0 \int_{B_r} q(\xi) \Phi^{\varepsilon_0}(\xi) \left(\frac{1}{|\xi|} - \frac{1}{r} \right) d\xi &= \int_{B_r} \Delta \Phi^{\varepsilon_0}(\xi) \left(\frac{1}{|\xi|} - \frac{1}{r} \right) d\xi = \\ &= -4\pi \Phi^{\varepsilon_0}(0) + \frac{1}{r^2} \int_{S_r} \Phi^{\varepsilon_0}(\omega) d\omega = \int_{S_1} \Phi^{\varepsilon_0}(r\theta) d\theta =: U(r). \end{aligned}$$

From the other hand, we have

$$\begin{aligned} \left| \int_{B_r} q(\xi) \Phi^{\varepsilon_0}(\xi) \left(\frac{1}{|\xi|} - \frac{1}{r} \right) d\xi \right| &\leq \max_{B_r} |q| \int_0^r d\tau \tau^2 \left(\frac{1}{\tau} - \frac{1}{r} \right) \int_{S_1} \Phi^{\varepsilon_0}(\tau\theta) d\theta \\ &\leq \max_{B_r} |q| r \int_0^r U(\tau) d\tau \leq \left[\max_{B_r} |q| \right] r^2 \left[\max_{[0,r]} U \right] \end{aligned}$$

and arrive at the estimate

$$U(r) \leq cr^2 \max_{[0,r]} U(\cdot).$$

Choosing r small enough such that $U(r) = \max_{[0,r]} U(\cdot)$ we see that this estimate is possible only if $U(r_1)$ vanishes identically for $r_1 < r$. Since $\Phi^{\varepsilon_0} \geq 0$, this means that $\Phi^{\varepsilon_0} \equiv 0$ in a small ball B_r .

So, Φ^{ε_0} is a solution of an elliptic equation vanishing in a ball. By the Landis uniqueness theorem [4], such a solution vanishes everywhere and we have $\Phi^{\varepsilon_0} = 0$ in \mathbb{R}^3 that contradicts to (2.9). \square

⁶in fact, analytically

3 s -points and degeneration of jets

3.1 Jets and polynomials

Recall what a *jet* is. For a fixed $a \in \mathbb{R}^3$ and an integer $k \geq 0$, we say the smooth functions u and v to be equivalent (and write $u \sim v$) if $u(x) - v(x) = o(|x - a|^k)$ as $x \rightarrow a$. With respect to the relation \sim , the set $C^\infty(\mathbb{R}^3)$ decays on the equivalence classes, whereas the class $j_a^k[u] := \{v \mid v \sim u\}$ is called the jet of u (at the point a , of the order k). Setting up $\alpha j_a^k[u] + \beta j_a^k[u] := j_a^k[\alpha u + \beta v]$ for $\alpha, \beta \in \mathbb{R}$, one makes the set of jets into a linear space $\mathcal{J}_a^k[C^\infty(\mathbb{R}^3)]$.

As is easy to recognize, the jet $j_a^k[u]$ can be identified with the collection of the derivatives $\{D_x^j u(a)\}_{|j|=0}^k$, whereas $\dim \mathcal{J}_a^k[C^\infty(\mathbb{R}^3)]$ is equal to the number d^k of such derivatives: $d^0 = 1$, $d^1 = 1 + 3 = 4$, $d^2 = 1 + 3 + 6 = 10$, \dots , $d^k = \frac{1}{6}(k+1)(k+2)(k+3)$.

Let $\mathcal{L} \subset C^\infty(\mathbb{R}^3)$ be a linear set; introduce a subspace

$$\mathcal{J}_a^k[\mathcal{L}] := \{j_a^k[u] \mid u \in \mathcal{L}\} \subset \mathcal{J}_a^k[C^\infty(\mathbb{R}^3)]$$

and denote by $d_a^k[\mathcal{L}]$ its dimension.

Fix an integer $l \geq 0$; the functions belonging to a linear set

$$\mathcal{P}_q^l := \left\{ p \in C^\infty(\mathbb{R}^3) \mid (-\Delta + q)p = 0 \text{ in } \mathbb{R}^3, \ |p(x)| \leq \text{const}(1 + |x|)^l \right\}.$$

are said to be q -harmonic polynomials of the order $\leq l$. The existence of such functions is established in a standard way: one puts $p = p_l + w$, where p_l is a harmonic polynomial (see, e.g., [9]), derives the relevant integral equation for w of the form analogous to (2.1), and proves its solvability and uniqueness of the solution by the same arguments as for the equation (2.1), i.e., referring to Convention 1. For instance, the function Φ determined by (2.4), (2.5) is an element of \mathcal{P}_q^0 . Also, as is easy to verify, $\mathcal{P}_q^l \subset \mathcal{P}_q^{l'}$ for $l < l'$, and the relations

$$\dim \mathcal{P}_q^l = \dim \mathcal{P}_0^l = (l+1)^2, \quad d_a^k[\mathcal{P}_q^k] \leq d_a^k[\mathcal{P}_0^k] = (k+1)^2 \quad (3.1)$$

hold.

The main result of this section is

Theorem 3 *For $m = 2, 3, \dots$, the inclusion $a \in \Upsilon_q^m$ is equivalent to the relation*

$$d_a^{2m-2}[\mathcal{P}_q^{2m-2}] < (2m-1)^2. \quad (3.2)$$

Since in the generic case one has $d_a^{2m-2}[\mathcal{P}_q^{2m-2}] = (2m-1)^2$ (see (3.1) for $k = 2m-2$), it is reasonable to refer to (3.2) as a jet degeneration of \mathcal{P}_q^{2m-2} at the point $x = a$. Also, the Theorem 2 can be interpreted in the same terms: $\Phi(a) = 0$ means that $\dim \mathcal{J}_a^0[\mathcal{P}_q^0] = 0 < 1$.

The proof is postponed till section 3.3. For the sake of simplicity, it will be demonstrated for the case $m = 2$; the way to treat the general case will be clear.

3.2 Green function

Here we deal with the q -biharmonic equation $(-\Delta + q)^2 u = h$. For $x, y \in \mathbb{R}^3$, $\langle x, y \rangle := x_1 y_1 + x_2 y_2 + x_3 y_3$ is the inner product.

Unperturbed case For $q = 0$, we set

$$B_0(x, y) := -\frac{|x - y|}{8\pi}$$

and, in view of the well-known relation

$$(-\Delta)^2 B_0(\cdot, y) = \delta_y, \quad (3.3)$$

refer to B_0 as the *unperturbed Green function*. The following is a standard way to derive the asymptotic of B_0 as $|x| \rightarrow \infty$:

$$\begin{aligned} |x - y| &= (|x|^2 - 2\langle x, y \rangle + |y|^2)^{\frac{1}{2}} = \\ |x| \left(1 + \frac{|y|^2}{|x|^2} - \frac{2}{|x|} \left\langle \frac{x}{|x|}, y \right\rangle \right)^{\frac{1}{2}} &= \\ |x| \left(1 + \frac{1}{2} \left(\frac{|y|^2}{|x|^2} - \frac{2}{|x|} \left\langle \frac{x}{|x|}, y \right\rangle \right) - \frac{1}{8} \left(\frac{|y|^2}{|x|^2} - \frac{2}{|x|} \left\langle \frac{x}{|x|}, y \right\rangle \right)^2 + O(|x|^{-3}) \right) &= \\ |x| \left(1 + \frac{|y|^2}{2|x|^2} - \frac{1}{|x|} \left\langle \frac{x}{|x|}, y \right\rangle - \frac{1}{2|x|^2} \left\langle \frac{x}{|x|}, y \right\rangle^2 + O(|x|^{-3}) \right) &= \\ |x| + \frac{|y|^2}{2|x|} - \left\langle \frac{x}{|x|}, y \right\rangle - \frac{1}{2|x|} \left\langle \frac{x}{|x|}, y \right\rangle^2 + O(|x|^{-2}) &= \\ |x| + \frac{|y|^2}{2|x|} - \frac{1}{|x|} \sum_i x_i y_i - \frac{1}{|x|^3} \sum_{i \neq j} x_i x_j y_i y_j - \frac{1}{2|x|^3} \sum_i x_i^2 y_i^2 + \\ + O\left(\frac{1}{|x|^2}\right), \end{aligned}$$

where i and j run over 1,2,3. To clarify the structure of this expression, one can use the identity

$$\sum_i x_i^2 y_i^2 = \frac{1}{2}(x_1^2 - x_2^2)(y_1^2 - y_2^2) + \frac{1}{6}(x_1^2 + x_2^2 - 2x_3^2)(y_1^2 + y_2^2 - 2y_3^2) + \frac{1}{3}|x|^2 |y|^2$$

and write the result in the form

$$\begin{aligned} |x - y| &= \left\{ |x| + \frac{|y|^2}{3|x|} \right\} - \left\{ \sum_i \left[\frac{x_i}{|x|} \right] y_i \right\} - \frac{1}{|x|} \left\{ \sum_{i \neq j} \left[\frac{x_i x_j}{|x|^2} \right] y_i y_j - \right. \\ &\quad \left. \frac{1}{4} \left[\frac{x_1^2 - x_2^2}{|x|^2} \right] (y_1^2 - y_2^2) - \frac{1}{12} \left[\frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^2} \right] (y_1^2 + y_2^2 - 2y_3^2) \right\} \\ &\quad + O\left(\frac{1}{|x|^2}\right). \end{aligned} \tag{3.4}$$

As is seen now, this is an expansion over the spherical harmonics: the terms into the first curly braces do not depend on angle variables, the ones in the second and third braces are proportional to $Y_1^m(\frac{x}{|x|})$ and $Y_2^m(\frac{x}{|x|})$ respectively⁷. One more peculiarity of such a representation is that among the y -dependent coefficients

$$\begin{aligned} \Phi_1^0 &= 1, \quad \Psi^0 = |y|^2, \\ \Phi_2^0 &= y_1, \quad \Phi_3^0 = y_2, \quad \Phi_4^0 = y_3, \\ \Phi_5^0 &= y_1 y_2, \quad \Phi_6^0 = y_2 y_3, \quad \Phi_7^0 = y_1 y_3, \quad \Phi_8^0 = y_1^2 - y_2^2, \quad \Phi_9^0 = y_1^2 + y_2^2 - 2y_3^2 \end{aligned} \tag{3.5}$$

there are nine *harmonic* functions $\Phi_1^0, \dots, \Phi_9^0$ and one *biharmonic* Ψ^0 .

Remark 1 *So, the leading part of the unperturbed Green function asymptotic (3.4) is of the following structure:*

- the term harmonic w.r.t. x has the coefficient Ψ^0 biharmonic w.r.t. y
- the other terms are biharmonic w.r.t. x and have the coefficients Φ_i^0 harmonic w.r.t. y .

⁷the harmonics are selected by the square brackets

Quite analogous picture will be observed in the perturbed case.

Perturbed case For $q \neq 0$, we *define* the (perturbed) Green function as a $L_\infty^{\text{loc}}(\mathbb{R}^3)$ -solution of the equation

$$B(x, y) = -\frac{|x - y|}{8\pi} + \int_{\mathbb{R}^3} \frac{|x - t|}{8\pi} LB(t, y) dt, \quad x \in \mathbb{R}^3, \quad (3.6)$$

where $y \in \mathbb{R}^3$ is a parameter and $L := (-\Delta + q)^2 - (-\Delta)^2$ is a second-order differential operator⁸ such that $\text{supp } Lu \subset \text{supp } q \cap \text{supp } u$. The following result shows that this definition is correct.

Lemma 2 *The equation (3.6) is uniquely solvable.*

Proof Let the homogeneous equation

$$\psi(x) = \int_{\mathbb{R}^3} \frac{|x - t|}{8\pi} L\psi(t) dt$$

have a nonzero solution ψ . Such a solution satisfies

$$(-\Delta + q)^2 \psi = 0 \quad \text{in } \mathbb{R}^3;$$

by (3.4), for large $|x|$ it behaves as

$$\psi(x) \underset{|x| \rightarrow \infty}{=} \alpha|x| + \sum_{i=1}^3 \beta_i \frac{x_i}{|x|} + \tilde{\psi}(x),$$

where $\tilde{\psi} = O(\frac{1}{|x|}) \rightarrow 0$.

Assume that ψ is q -harmonic: $(-\Delta + q)\psi = 0$ in \mathbb{R}^3 . If so, we have $\Delta\psi = 0$ for large $|x|$ that implies $\alpha = \beta_i = 0$ and $\psi = \tilde{\psi}$. Hence, ψ is a q -harmonic function vanishing as $|x| \rightarrow \infty$ that is forbidden by Convention 1.

Thus, $\varphi := (-\Delta + q)\psi$ is a nonzero function, which satisfies $(-\Delta + q)\varphi = 0$ in \mathbb{R}^3 and

$$\varphi(x) \underset{|x| \rightarrow \infty}{=} -\alpha\Delta|x| - \sum_{i=1}^3 \beta_i \Delta \frac{x_i}{|x|} + \Delta\tilde{\psi}(x) = O\left(\frac{1}{|x|}\right) \rightarrow 0$$

⁸in the integral, L acts on the variable t

that is impossible in view of Convention 1. So, assuming $\psi \neq 0$ we arrive at a contradiction. \square

As easily follows from (3.6), the relation

$$(-\Delta + q)^2 B(\cdot, y) = \delta_y \quad (3.7)$$

analogous to (3.3) is valid. Also, the symmetry property $B(x, y) = B(y, x)$ holds.

The way to derive the asymptotic of B analogous to (3.4) is also quite standard. Namely, we represent

$$B(x, y) = -\frac{1}{8\pi} A(x, y) + O(|x|^{-2}) \quad (3.8)$$

with an ansatz of the form⁹

$$\begin{aligned} A(x, y) = & \left\{ |x| \Phi_1(y) + \frac{\Psi(y)}{3|x|} \right\} - \\ & \left\{ \left[\frac{x_1}{|x|} \right] \Phi_2(y) + \left[\frac{x_2}{|x|} \right] \Phi_3(y) + \left[\frac{x_3}{|x|} \right] \Phi_4(y) \right\} - \\ & \frac{1}{|x|} \left\{ \left[\frac{x_1 x_2}{|x|^2} \right] \Phi_5(y) + \left[\frac{x_2 x_3}{|x|^2} \right] \Phi_6(y) + \left[\frac{x_1 x_3}{|x|^2} \right] \Phi_7(y) + \right. \\ & \left. \frac{1}{4} \left[\frac{x_1^2 - x_2^2}{|x|^2} \right] \Phi_8(y) - \frac{1}{12} \left[\frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^2} \right] \Phi_9(y) \right\}, \end{aligned} \quad (3.9)$$

tend $|y| \rightarrow \infty$ in (3.6), using the symmetry property substitute this representation to (3.6), and by equaling the coefficients at the proper spherical harmonics, eventually get the 'transport equations' of the form

$$\Phi_i(x) = \Phi_i^0(x) + \int_{\mathbb{R}^3} \frac{|x-t|}{8\pi} L\Phi_i(t) dt, \quad i = 1, \dots, 9 \quad (3.10)$$

$$\Psi(x) = \Psi^0(x) + \int_{\mathbb{R}^3} \frac{|x-t|}{8\pi} L\Psi(t) dt. \quad (3.11)$$

By the same arguments, which provide the unique solvability of (3.6), these equations are also solvable uniquely.

As is easy to see from (3.10) and (3.11), the solutions Ψ and Φ_i are q-biharmonic. Moreover,

⁹compare with (3.4)!

Lemma 3 *The functions Φ_1, \dots, Φ_9 are q -harmonic polynomials of the order ≤ 2 and constitute a basis in \mathcal{P}_q^2 .*

Proof

Step 1: q -harmonicity Take a $f \in C_0^\infty(\mathbb{R}^3)$, denote $\hat{H} := (-\Delta + q)^{10}$ and put

$$F(x) := \int_{\mathbb{R}^3} B(x, y) \hat{H}f(y) dy, \quad x \in \mathbb{R}^3. \quad (3.12)$$

Applying \hat{H}^2 and taking into account (3.7), we get $\hat{H}^2 F = \hat{H}f$ and, hence,

$$\hat{H}(\hat{H}F - f) = 0 \quad \text{in } \mathbb{R}^3.$$

As is easily seen from (3.12) and (3.8), (3.9), the function $\hat{H}F$ vanishes as $|x| \rightarrow \infty$. Hence, the same is valid for $\hat{H}F - f$. In the mean time, the latter difference is annihilated by \hat{H} and, whence, it has to vanish identically by Convention 1. Thus, we arrive at $\hat{H}F - f = 0$, i.e.,

$$(-\Delta + q(x)) \int_{\mathbb{R}^3} B(x, y) \hat{H}f(y) dy = f(x), \quad x \in \mathbb{R}^3. \quad (3.13)$$

Now, tending $|x| \rightarrow \infty$ and taking into account the structure of the asymptotic (3.8), (3.9), it is easy to see that the asymptotic of the l.h.s. contains certain linearly independent harmonic (w.r.t. x) terms¹¹ with the coefficients of the form $\int_{\mathbb{R}^3} \Phi_i(y) \hat{H}f(y) dy$. In the mean time, the r.h.s. in the equality (3.13) is compactly supported and, hence, the equality forces all harmonic terms to vanish identically. This follows to

$$0 = \int_{\mathbb{R}^3} \Phi_i(y) \hat{H}f(y) dy = \int_{\mathbb{R}^3} \hat{H}\Phi_i(y) f(y) dy$$

that is equivalent to $\hat{H}\Phi_i = 0$ in \mathbb{R}^3 by arbitrariness of f . So, Φ_i are q -harmonic.

Step 2: integral equations The functions Φ_i satisfy the integral equations, which are different from and more informative than (3.10). These equations can be derived as follows.

¹⁰ So, \hat{H} is understood as a differential expression that can be applied to any smooth enough function not necessarily belonging to \mathcal{H} .

¹¹for large enough $|x|$, 'harmonic' and 'q-harmonic' is the same

By (2.2), for any $f \in C_0^\infty(\mathbb{R}^3)$ one has the relation

$$(-\Delta + q(x)) \int_{\mathbb{R}^3} G(x, y) f(y) dy = f(x), \quad x \in \mathbb{R}^3.$$

Comparing it with (3.13) we get

$$\int_{\mathbb{R}^3} G(x, y) f(y) dy = \int_{\mathbb{R}^3} B(x, y) \hat{H} f(y) dy,$$

whereas integration by parts easily implies

$$(-\Delta + q(y)) B(x, y) = G(x, y), \quad x, y \in \mathbb{R}^3, \quad x \neq y. \quad (3.14)$$

In the unperturbed case, for the Green functions B_0 and G_0 we have just

$$-\Delta \left[-\frac{|x-y|}{8\pi} \right] = \frac{1}{4\pi|x-y|}, \quad x, y \in \mathbb{R}^3, \quad x \neq y. \quad (3.15)$$

These relations enable one to get the asymptotics for G and G_0 through (3.8) and (3.4) respectively, *the coefficients of the asymptotics being expressed through Φ_i, Ψ and Φ_i^0, Ψ^0* . Surely, the asymptotic of G obtained in this way is just a more detailed version of (2.3), which takes into account the structure of the lower order terms. Also, the comparison of the coefficients at $\frac{1}{|x|}$ in the detailed asymptotic and (2.3) easily implies the important relation

$$\Phi(y) = -\frac{1}{6} (\Delta + q(y)) \Psi(y), \quad (3.16)$$

which will be required later on.

Thereafter, the trick, which was used for derivation of the equation (2.6), is repeated: substituting the detailed asymptotics of G and G_0 to (2.1) and comparing the terms in the left and right hand sides, we can arrive at the equations

$$\Phi_i(x) = \Phi_i^0(x) - \int_{\mathbb{R}^3} \frac{q(s)}{4\pi|x-s|} \Phi_i(s) ds, \quad x \in \mathbb{R}^3, \quad i = 1, \dots, 9, \quad (3.17)$$

the first one being identical to (2.6).

Step 3: completing the proof The estimates $|\Phi_i(x)| \leq c(1+|x|)^2$ easily follow from the integral equations (3.17) and the evident estimates of the same form for Φ_i^0 . Thus, we have $\Phi_i \in \mathcal{P}_q^2$.

As is well known, the polynomials $\Phi_1^0, \dots, \Phi_9^0$ form a basis in (unperturbed) \mathcal{P}_0^2 . Using the equations (3.17), it is not difficult to conclude that the same is valid for Φ_1, \dots, Φ_9 in the perturbed \mathcal{P}_q^2 . \square

3.3 Proving Theorem 3

Recall that we deal with the case $m = 2$;

Necessity Assume that $a \in \Upsilon_q^2$ holds. It means that there is a function $h \in \mathcal{H}$ such that $(-\Delta + q)^2 h = 0$ in $\mathbb{R}^3 \setminus \{a\}$ and, in the same time, $(-\Delta + q)h$ does not vanish identically.

Considering $(-\Delta + q)^2 h$ as a distribution on $C_0^\infty(\mathbb{R}^3)$, we see that it is supported at $x = a$. Such a distribution has to be a linear combination of the Dirac measure derivatives:

$$(-\Delta + q)^2 h = \sum_{|j|=0}^N \alpha_j D_x^j \delta_a = \sum_{|j|=0}^N \alpha_j (-1)^{|j|} D_a^j \delta_a =: P_N(\nabla_a) \delta_a,$$

where $P_N(\nabla_a)$ is a differential polynomial of the order N with constant coefficients, which acts on the variable a . Therefore, in accordance with (3.7), the function h has to be of the form

$$h(x) = P_N(\nabla_a) B(x, a) + \tilde{h}(x), \quad x \in \mathbb{R}^3 \setminus \{a\}$$

with a smooth q -biharmonic in \mathbb{R}^3 function \tilde{h} .

Since

$$B(x, a) = c|x - a| + \text{smoother terms}$$

near $x = a$ (see (3.6)), the only option for h to belong to \mathcal{H} (to be square-summable near $x = a$) is $\alpha_j = 0$ for $|j| \geq 3$; hence, we have $N \leq 2$, i.e.,

$$h(x) = P_2(\nabla_a) B(x, a) + \tilde{h}(x), \quad x \in \mathbb{R}^3 \setminus \{a\}.$$

Let us show that $\tilde{h} = 0$. At first, note that for $|x| \rightarrow \infty$ the function \tilde{h} grows *not slower* than $|x|^2$. Indeed, otherwise it grows not faster than $|x|$ and, by the latter, the function $\varphi := (-\Delta + q)\tilde{h}$ is q -harmonic in \mathbb{R}^3 and tends to zero that is impossible by Convention 1¹². On the other hand, the summand $P_2(\nabla_a) B(x, a)$ may grow *not faster* than $|x|$, whereas the sum h belongs to \mathcal{H} . Hence, \tilde{h} must vanish identically in \mathbb{R}^3 . Thus, we get

$$h(x) = P_2(\nabla_a) B(x, a), \quad x \in \mathbb{R}^3 \setminus \{a\}. \quad (3.18)$$

¹²The possibility $\varphi \equiv 0$ is also excluded by the same arguments as were used in the proof of Lemma 2.

Recall that $h \in \mathcal{H}$ and tend $|x| \rightarrow \infty$. Turning to the structure of the asymptotic of B (see (3.8), (3.9) with $y = a$) and taking into account the linear independence of the spherical harmonics (the square brackets in (3.9)), we easily conclude that the r.h.s. of (3.18) can be square-summable if and only if the coefficients at the terms, which behave as $|x|$ and $\frac{1}{|x|}$, vanish, i.e.,

$$P_2(\nabla_a) \Phi_i(a) = 0, \quad i = 1, \dots, 9; \quad P_2(\nabla_a) \Psi(a) = 0. \quad (3.19)$$

The obtained relations for q -harmonic Φ_i are nontrivial only if $P_2(\nabla_a) \neq -\Delta + q(a)$. Show that it is the case. Indeed, otherwise, we have

$$\Phi(a) = \langle \text{see (3.16)} \rangle = -\frac{1}{6} P_2(\nabla) \Psi(a) = \langle \text{see (3.19)} \rangle = 0.$$

By (2.3), we conclude that $G(\cdot, a) \in \mathcal{H}$, i.e., a is a s -point of the *first* order, i.e., $a \in \Upsilon_q^1$ that contradicts to the assumption.

Since Φ_1, \dots, Φ_9 constitute a basis in \mathcal{P}_q^2 , any q -harmonic polynomial $p \in \mathcal{P}_q^2$ satisfies $P_2(\nabla)p(a) = 0$. Hence, $d_a^2[\mathcal{P}_q^2] \leq 8$, i.e., the jet degeneration at the point a does occur.

Sufficiency Assume that at a point $a \in \mathbb{R}^3$ the jet degeneration occurs, i.e., $d_a^2[\mathcal{P}_q^2] < 9$. Let \mathcal{L} be the 10-dimensional subspace spanned on $\Phi_1, \dots, \Phi_9, \Psi$; the degeneration evidently implies $d_a^2[\mathcal{L}] < 10$. By the latter, there exists a second order differential polynomial $P_2(\nabla)$ such that (3.19) is valid. In the mean time, the same arguments as in the item below (3.19) imply $P_2(\nabla) \neq -\Delta + q(a)$.

Defining a function h by the r.h.s. of (3.18) and taking into account the form of the asymptotic (3.8) and the relations (3.19), it is easy to see that $h \in \mathcal{H}$ and $(-\Delta + q)^2 h = 0$ in $\mathbb{R}^3 \setminus \{a\}$. Hence, $a \in \Upsilon_q^2$ holds. \square

In the case of $a \in \Upsilon_q^j$ for $j > 2$, the proof is just more complicated in notation. However, it exploits the same idea: a linear combination of the relevant q - j -harmonic Green function and its derivatives can belong to the space \mathcal{H} iff the proper analog of the relations (3.19) does hold that is equivalent to a certain jet degeneration at $x = a$.

3.4 Radially symmetric potential

If the potential is of the form $q = q(|x|)$, much more can be said about structure of the s -point set Υ_q . As is evident, Υ_q consists of spheres in \mathbb{R}^3 centered at $x = 0$. Here we briefly announce some results on this case.

1. For a fixed $l \geq 0$, the set Υ_q^l can be characterized as follows. Let φ_l be a *regular* solution of the radial Schrödinger equation

$$-\varphi_l'' + \frac{l(l+1)}{r^2} \varphi_l + q(r)\varphi_l = 0, \quad r > 0 \quad (3.20)$$

that behaves as $\varphi_l(r) \sim r^{l+1}$ near $r = 0$. Note that, in this case, the functions

$$\frac{\varphi_l(|x|)}{|x|} Y_l^m \left(\frac{x}{|x|} \right) \quad (|m| \leq l+1)$$

are q -harmonic and belong to \mathcal{P}_q^l . Introduce the Kram determinants¹³

$$\Delta_l^m(r) := \begin{vmatrix} \varphi_m(r) & \varphi'_m(r) & \cdots & \varphi_m^{(l-m)}(r) \\ \varphi_{m+1}(r) & \varphi'_{m+1}(r) & \cdots & \varphi_{m+1}^{(l-m)}(r) \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_l(r) & \varphi'_l(r) & \cdots & \varphi_l^{(l-m)}(r) \end{vmatrix}, \quad m = 0, 1, \dots, l.$$

Let N_j be the number of the eigenvalues of the partial Schrödinger operator $-d_r^2 + \frac{j(j+1)}{r^2} + q(r)$ in $L_2(0, \infty)$ and $z(\Delta_l^m)$ the number of zeros of the Kram determinant. The relation

$$z(\Delta_l^m) = \sum_{k=0}^{l-m} (-1)^k N_{m+k} \quad (3.21)$$

is valid.

2. As we saw in section 2, $\sigma_{\text{disc}}(H) \neq \emptyset$ implies $\Upsilon_q \neq \emptyset$. Our hypothesis is that the converse is also valid, so that the equivalence

$$\{\sigma_{\text{disc}}(H) = \emptyset\} \Leftrightarrow \{\Upsilon_q \neq \emptyset\} \quad (3.22)$$

holds. It can be shown that *for radial symmetric potentials, if $\sigma_{\text{disc}}(H) = \emptyset$ then $\Upsilon_q^k = \emptyset$ holds for all $k \geq 1$* . This fact is established in the following way.

One can show that $a \in \mathbb{R}^3$ is a l -jet degeneration point of the q -harmonic polynomials iff a certain $(l+1) \times (l+1)$ -determinant $\delta_l(r)$

¹³It is worthy of noting that this determinants appear in Darboux transform theory

vanishes as $r = |a|$. In the mean time, the determinant can be represented in the form

$$\delta_l(r) = \Delta_l^0 (\Delta_l^1(r))^2 \dots (\Delta_l^l(r))^2$$

that enables one to find the zeros of δ_l by the use of (3.21) and, hence, to verify the presence/absence of the jet degeneration points.

3. The picture of s -spheres is rather curious. For instance, in the case of $H_\alpha = -\Delta + \alpha q$ with an interaction constant $\alpha \geq 0$, the following scenario is possible as α grows:

(a) for small enough $\alpha \in [0, \alpha_1]$, one has $\sigma_{\text{disc}}(H_\alpha) = \emptyset$ and $\Upsilon_q^k = \emptyset$ for all $k \geq 1$

(b) for $\alpha \in [\alpha_1, \alpha_2]$, $\sigma_{\text{disc}}(H_\alpha)$ consists of a single eigenvalue $\lambda_1^\alpha < 0$, the eigenfunction χ_1^α being radially symmetric, whereas all Υ_q^k , $k \geq 1$ are nonempty and consist of the spheres S_k of the radius r_k growing to infinity as $k \rightarrow \infty$

(c) Assume that as α passes through α_2 , the second negative eigenvalue $\lambda_2^\alpha > \lambda_1^\alpha$ appears, the eigenfunction χ_2^α depending on angle variable as $Y_1(\cdot)$. In such a case, as $\alpha \uparrow \alpha_2$, the certain spheres S_k , $k \geq 2$ are blown up: $\lim_{\alpha \rightarrow \alpha_2} r_k = \infty$.

In the mean time, if λ_2^α is such that χ_2^α is also radially symmetric, the picture is quite different: all Υ_q^k , $k \geq 1$ are nonempty and evolve regularly as α grows.

3.5 Comments

- Some results of the book [1] on a system of the Schrödinger equations (3.20) with different $l \geq 0$ can be interpreted in 's-point terms'. In particular, the existence of the zero energy solution of the system, which is bounded at $r = 0$, leads to a certain 'non-standard' factorization of the S -matrix that is quite analogous to the effects discussed in sec.1.
- Let $x = a$ be a s -point of the potential q , and w_a^f a reversing wave, which satisfies

$$w_a^f|_{|x-a|<|t|} = 0. \quad (3.23)$$

The Fourier transform $\tilde{w}_a(\cdot, k) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{ikt} w_a(\cdot, t) dt$ obeys

$$(-\Delta + q)\tilde{w}_a = k^2\tilde{w}_a,$$

so that $\tilde{w}_a(\cdot, k)$ is a continuous spectrum eigenfunction of the Schrödinger operator H . A remarkable fact is that, by (3.23), such an eigenfunction is an *entire* function of the spectral parameter k . In due time, the existence of such eigenfunctions was a question, which has been affirmatively answered by R.Newton. Now we see that there exists a rich family of entire continuous spectrum eigenfunctions $\{\tilde{w}_a(\cdot, k)\}_{a \in \Upsilon_q}$ associated with *reversing finite energy waves* and parametrized by points of the surfaces, which Υ_q consists of. The concrete examples demonstrate that these surfaces may be of rather complicated shape (ovaloids, toruses, etc). The meaning and role of such eigenfunctions in the scattering theory are not quite clear yet.

- The important question whether the set of the finite order s -points $\cup_{m=1}^{\infty} \Upsilon_q^m$ exhausts Υ_q is still open even in the case of radially symmetric potentials. Also, the conjecture (3.22) is not justified yet.

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